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Problems of cuts in a composite elastic wedge☆

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ABSTRACT

Problems of strip and elliptical cuts (tensile cracks) in the middle of a three-layer elastic wedge are investigated in a three-dimensional formulation. Free or rigid clamping conditions or the stress-free condition are stipulated on the outer surfaces of the composite wedge. The problems are assumed to be symmetrical about the plane of the cut. The wedge-shaped layer containing the cut is incompressible and hinged along both faces with two other layers. The integral equations of the problems with respect to the opening of the cut are derived. Inverse operators are obtained for the operators occurring in the kernels of these equations. The relation between problems on cuts and the corresponding contact problems for a composite wedge of half the aperture angle is used. The method of paired integral equations is used for the case of a strip cut emerging from the edge of the wedge. The problems are reduced to Fredholm integral equations of the second kind in certain auxiliary functions, in terms of the values of which the normal stress intensity factors are expressed. A regular asymptotic solution is constructed for the case of an elliptic cut.

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Similar three-dimensional problems on strip and elliptic cuts in a uniform elastic wedge were considered previously in Refs 1 and 2, as well as problems of a crack in a plane wedge.[3]

1. Formulation of the problems

We will consider, in cylindrical coordinates r, φ, z, a three-dimensional wedge consisting of layers

$$\Omega_1 = \{r\in[0,\infty),\ \varphi\in[-(\alpha+\beta),-\beta],\ z\in(-\infty,\infty)\}$$
$$\Omega_2 = \{r\in[0,\infty),\ \varphi\in[-\beta,\beta],\ z\in(-\infty,\infty)\}$$
$$\Omega_3 = \{r\in[0,\infty),\ \varphi\in[\beta,\alpha+\beta],\ z\in(-\infty,\infty)\}$$

hinged along the edges $\varphi=\pm\beta$. The outer edges $\varphi=\pm(\beta+\alpha)$ are hinged or rigidly clamped or are stress free (Problems A, B and C respectively). The layers Ω_1 and Ω_3 have the elasticity parameters G_1 (the shear modulus) and ν_1 (Poisson's ratio), while the layer Ω_2 has the parameters G and $\nu=0.5$ (incompressible material). In the middle of the half-plane of the composite wedge $\varphi=0$ there is a cut in the region $(r,z)\in\Omega$, which is in the open state under a normal load $\sigma_\varphi=-q(r,z)$, $(r,z)\in\Omega$, $\varphi=\pm0$. It is required to determine the value of the opening of the cut $u_\varphi=\pm f(r,z)$, $(r,z)\in\Omega$, $\varphi=\pm0$. The normal stress intensity factor can then be found. Using the symmetry of the problems about the plane of the cut, we will further consider the region $0\le\varphi\le\alpha+\beta$, the boundary conditions in which we will write in the form (we will use the superscript 1 for displacements and stresses in the region Ω_3)

$$\varphi=0:\ \tau_{r\varphi}=\tau_{\varphi z}=0\ \forall(r,z);\ u_\varphi=0\ (r,z)\notin\Omega,\quad \sigma_\varphi=-q(r,z)\ (r,z)\in\Omega$$
$$\varphi=\beta:\ u_\varphi=u_\varphi^1,\quad \sigma_\varphi=\sigma_\varphi^1,\quad \tau_{r\varphi}=\tau_{r\varphi}^1=\tau_{\varphi z}=\tau_{\varphi z}^1=0$$
$$\varphi=\alpha+\beta:\ \text{A)}\ u_\varphi^1=\tau_{r\varphi}^1=\tau_{\varphi z}^1=0,\ \text{B)}\ u_r^1=u_\varphi^1=u_z^1=0,\ \text{C)}\ \sigma_\varphi^1=\tau_{r\varphi}^1=\tau_{\varphi z}^1=0 \tag{1.1}$$

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Using the technique of reducing the problem of the theory of elasticity to a Vekua generalized Hilbert boundary-value problem,[4,5] and complex Fourier and Kontorovich–Lebedev integral transformations, we reduce boundary-value problems (1.1) to the following integral equation in the function $f(r, z)$

$$\int_\Omega f(x, y)K(x, y, r, z)dxdy = \frac{\pi}{G}q(r, z), \quad (r, z) \in \Omega$$

$$K(x, y, r, z) = \frac{4}{\pi^2 rx}\int_0^\infty\int_0^\infty \tau \operatorname{sh}\pi\tau K_{i\tau}(tr)A_\tau\{sK_{is}(tx)\}\cos t(z-y)d\tau dt \tag{1.2}$$

where $K_{i\tau}(r)$ is the Bessel function.

The operator A_τ for Problem A is given by the formulae

$$A_\tau\{sK_{is}(tx)\} = \tau K_{i\tau}(tx)W(\tau) + \frac{8(1-\nu_1)\varepsilon W_0(\tau)}{\operatorname{ch}(\pi\tau/2)W_1(\tau)}F(\tau)$$

$$W(\tau) = W_+^{(2\beta)}(\tau), \quad W_0(\tau) = \frac{\operatorname{sh}\beta\tau\cos\beta + \tau\operatorname{ch}\beta\tau\sin\beta}{\operatorname{ch}2\beta\tau - \cos 2\beta}, \quad \varepsilon = \frac{G}{G_1}$$

$$W_1(\tau) = W_+^{(2\alpha)}(\tau) + 2(1-\nu_1)\varepsilon W(\tau), \quad W_\pm^{(\delta)}(\tau) = \pm\frac{\operatorname{sh}\delta\tau \pm \tau\sin\delta}{\operatorname{ch}\delta\tau \mp \cos\delta} \tag{1.3}$$

The function $F(\tau)$ satisfies a Fredholm integral equation of the second kind

$$F(\tau) - (1-2\nu_1)\int_0^\infty L_1(\tau, u)F(u)du = -\operatorname{ch}\frac{\pi\tau}{2}W_0(\tau)\tau K_{i\tau}(tx), \quad 0 \le \tau < \infty$$

$$L_1(\tau, u) = 2\operatorname{ch}\frac{\pi\tau}{2}\operatorname{sh}\frac{\pi u}{2}\frac{1}{W_1(u)}\int_0^\infty h(\tau, u, w)g(w)dw$$

$$h(\tau, u, w) = \frac{\operatorname{sh}\pi w}{(\operatorname{ch}\pi w + \operatorname{ch}\pi\tau)(\operatorname{ch}\pi w + \operatorname{ch}\pi u)}, \quad g(w) = g_+^{(\alpha)}(w) = \frac{\operatorname{cth}\alpha w\sin^2 2\alpha}{\operatorname{ch}2\alpha w - \cos 4\alpha} \tag{1.4}$$

For Problem B in formulae (1.3) and (1.4) only the functions $W_1(\tau)$ and $g(w)$ are changed. In this case

$$W_1(\tau) = \frac{2\kappa\operatorname{ch}2\alpha\tau + 2\tau^2(1-\cos 2\alpha) + \kappa^2 + 1}{2\kappa\operatorname{sh}2\alpha\tau - 2\tau\sin 2\alpha} + 2(1-\nu_1)\varepsilon W(\tau), \quad \kappa = 3 - 4\nu_1$$

$$g(w) = -g_-^{(\alpha)}(w) + \{\sin^2\alpha[f_0(w)(2f_1(w) - wf_2(w)) - f_3(w)(2f_2(w) + wf_1(w))] -$$
$$-2(1-\nu_1)\sin\alpha[f_0(w)(\sin 3\alpha - \sin\alpha\operatorname{ch}2\alpha w) - f_3(w)\cos\alpha\operatorname{sh}2\alpha w]\}/f_4(w)$$

$$g_-^{(\alpha)}(w) = \frac{\operatorname{th}\alpha w\sin^2 2\alpha}{\operatorname{ch}2\alpha w + \cos 4\alpha}$$

$$f_0(w) = \kappa\operatorname{sh}2\alpha w\cos 2\alpha - w\sin 2\alpha, \quad f_1(w) = \cos 2\alpha + \sin^2 2\alpha - \operatorname{ch}2\alpha w$$
$$f_2(w) = \sin 2\alpha\operatorname{th}\alpha w(1+\cos 2\alpha), \quad f_3(w) = (\kappa\operatorname{ch}2\alpha w - 1)\sin 2\alpha$$

$$f_4(w) = [f_0^2(w) + f_3^2(w)](\operatorname{sh}^2\alpha w + \cos^2 2\alpha) \tag{1.5}$$

For Problem C

$$A_\tau\{sK_{is}(tx)\} = \tau K_{i\tau}(tx)W(\tau) + \frac{8(1-\nu_1)\varepsilon W_0(\tau)}{\operatorname{ch}(\pi\tau/2)}\left[\frac{F_+(\tau)}{W_+(\tau)} - \frac{F_-(\tau)}{W_-(\tau)}\right]$$

$$W_\pm(\tau) = W_\pm^{(\alpha)}(\tau) \pm (1-\nu_1)\varepsilon W(\tau) \tag{1.6}$$

The functions $F_{\pm}(\tau)$ are found from the system of Fredholm integral equations of the second kind ($0 \le \tau < \infty$)

$$F_{\pm}(\tau) - (1-2\nu_1)\int_0^\infty L_{\pm}(\tau,u)F_{\pm}(u)du = \pm\frac{(1-\nu_1)\varepsilon}{W_{\mp}(\tau)}W(\tau)F_{\mp}(\tau) - \frac{1}{2}\operatorname{ch}\frac{\pi\tau}{2}W_0(\tau)\tau K_{i\tau}(tx)$$

$$L_{\pm}(\tau,u) = 2\operatorname{ch}\frac{\pi\tau}{2}\operatorname{sh}\frac{\pi u}{2}\frac{1}{W_{\pm}(u)}\int_0^\infty h(\tau,u,w)g_{\pm}^{(\alpha/2)}(w)dw \tag{1.7}$$

Note that Problem A when $\alpha = \beta = \pi/2$ corresponds to the case of a cut in an incompressible elastic half-space, in contact, without friction, with an elastic half-space of another material.

2. A strip cut

Suppose the cut occupies the strip region

$$\Omega = \{r \in [0,a],\ z \in (-\infty,\infty)\}$$

emerging from the edge of the wedge. We will assume that the function $q(r, z)$ is periodic in z with period $2l$, expandable in a Fourier series and even in z. It is then sufficient to consider the case

$$q(r,z) = q(r)\cos(pz), \quad p = \pi n/l$$

and then to compose the superposition of solutions for different values of $n \ge 1$ when the solution for the limiting case of plane deformation ($n = 0$) is taken into account. In the problem considered, assuming $f(r, z) = f(r)\cos(pz)$ in integral Eq. (1.2), we reduce it to the following one-dimensional integral equation in the function $f(r)$

$$\int_0^a f(x)K(x,r)dx = \frac{q(r)}{G}, \quad 0 \le r \le a$$

$$K(x,r) = \frac{4}{\pi^2 rx}\int_0^\infty \tau\operatorname{sh}\pi\tau K_{i\tau}(pr)A_\tau\{sK_{is}(px)\}d\tau \tag{2.1}$$

We will use the method of paired integral equations[1,6] to solve Eq. (2.1). When deriving the paired integral equations a knowledge of the inverse operators to the operators A_τ of the form (1.3) and (1.6) is required. These operators are found by analysing the corresponding contact problems for a two-layer composite wedge of half the angular aperture, when the region of the cut is replaced a contact area.[7] When analysing contact problems the technique of reducing the problem of the theory of elasticity to a generalized Hilbert problem is also employed. The functional equations with a shift in contact problems are reduced to a Fredholm integral equation of the second kind, in terms of the solutions of which the required inverse operators acting in the space $\mathbf{C}_M(0, \infty)$ of functions, continuous and bounded on the semiaxis, are expressed. As a result the following two theorems are proved.

Theorem 1. *The operator $A_\tau\{d(s)\}$: $\mathbf{C}_M(0,\infty) \to \mathbf{C}_M(0,\infty)$ of the form (1.3) has an inverse, equal to $B_\tau = (A_\tau)^{-1}$, where*

$$B_\tau\{d(s)\} = \frac{d(\tau)}{W(\tau)} - B_\tau^*\{d(s)\}, \quad B_\tau^*\{d(s)\} = \frac{8(1-\nu_1)\varepsilon W_0(\tau)}{W(\tau)W_2(\tau)\operatorname{ch}(\pi\tau/2)}\Phi(\tau)$$

$$W_2(\tau) = W_+^{(2\alpha)}(\tau) + 2(1-\nu_1)\varepsilon W_3(\tau), \quad W_3(\tau) = \frac{\operatorname{ch}2\beta\tau - 1 - 2\tau^2\sin^2\beta}{\operatorname{sh}2\beta\tau + \tau\sin 2\beta} \tag{2.2}$$

The function $\Phi(\tau)$ satisfies the Fredholm integral equation of the second kind ($0 \le \tau < \infty$)

$$\Phi(\tau) - (1-2\nu_1)\int_0^\infty L_2(\tau,u)\Phi(u)du = -\operatorname{ch}\frac{\pi\tau}{2}\frac{W_0(\tau)}{W(\tau)}d(\tau)$$

$$L_2(\tau,u) = 2\operatorname{ch}\frac{\pi\tau}{2}\operatorname{sh}\frac{\pi u}{2}\frac{1}{W_2(u)}\int_0^\infty h(\tau,u,w)g(w)dw \tag{2.3}$$

Theorem 2. *The operator $A_\tau\{d(s)\}$: $\mathbf{C}_M(0,\infty) \to \mathbf{C}_M(0,\infty)$ of the form (1.6) has an inverse, equal to $B_\tau = (A_\tau)^{-1}$, where*

$$B_\tau\{d(s)\} = \frac{d(\tau)}{W(\tau)} - B_\tau^*\{d(s)\}, \quad B_\tau^*\{d(s)\} = \frac{8(1-\nu_1)\varepsilon W_0(\tau)}{\operatorname{ch}(\pi\tau/2)W(\tau)}\left[\frac{\Phi_+(\tau)}{W^+(\tau)} - \frac{\Phi_-(\tau)}{W^-(\tau)}\right]$$

$$W^{\pm}(\tau) = W_{\pm}^{(\alpha)}(\tau) \pm (1-\nu_1)\varepsilon W_3(\tau) \tag{2.4}$$

The functions $\Phi_\pm(\tau)$ are found from the system of Fredholm integral equations of the second kind ($0 \le \tau < \infty$)

$$\Phi_\pm(\tau) - (1-2\nu_1)\int_0^\infty L^\pm(\tau,u)\Phi_\pm(u)du = \pm\frac{(1-\nu_1)\varepsilon}{W^\mp(\tau)}W_3(\tau)\Phi_\mp(\tau) - \frac{1}{2}\operatorname{ch}\frac{\pi\tau}{2}\frac{W_0(\tau)}{W(\tau)}d(\tau)$$

$$L^\pm(\tau,u) = 2\operatorname{ch}\frac{\pi\tau}{2}\operatorname{sh}\frac{\pi u}{2}\frac{1}{W^\pm(u)}\int_0^\infty h(\tau,u,w)g_\pm(w)dw \tag{2.5}$$

Examples. When $\varepsilon = 0$ Theorems 1 and 2 obviously hold. We will verify Theorem 1 in another limiting case: $\nu_1 = 0.5$ and $\varepsilon \to \infty$. From relations (1.3) and (2.2) we have

$$A_\tau\{d(s)\} = d(\tau)W(\tau)\left[1 - 4\frac{W_0^2(\tau)}{W^2(\tau)}\right], \quad B_\tau\{d(s)\} = \frac{d(\tau)}{W(\tau)}\left[1 + \frac{4W_0^2(\tau)}{W(\tau)W_3(\tau)}\right] \tag{2.6}$$

It is easy to show that the expressions in square brackets are mutually inverse, i.e., Theorem 1 is satisfied.

Using Theorem 1 for Problems A and B and Theorem 2 for Problem C, we reduce Eq. (2.1) to the paired integral equations

$$\int_0^\infty Q(u)uW(u)K_{iu}(pr)du = \frac{\pi^2 rq(r)}{4G}, \quad 0 \le r < a$$

$$\int_0^\infty \operatorname{sh}\pi u B_u\left\{\frac{Q(s)W(s)}{\operatorname{sh}\pi s}\right\}K_{iu}(pr)du = 0, \quad a < r < \infty \tag{2.7}$$

in the function $Q(u)$, which is related to the required function as follows:

$$f(r) = \frac{2}{\pi^2}\int_0^\infty \operatorname{sh}\pi u B_u\left\{\frac{Q(s)W(s)}{\operatorname{sh}\pi s}\right\}K_{iu}(pr)du \tag{2.8}$$

Looking for a solution of integral Eq. (2.7) in the form

$$Q(u) = Q_1(u) + Q_2(u)$$

$$Q_1(u) = \frac{\operatorname{sh}\pi u}{2GW(u)}\int_0^a q(r)K_{iu}(pr)dr, \quad Q_2(u) = -\frac{2\sqrt{2}}{\pi\sqrt{\pi}}\frac{\operatorname{sh}\pi u}{W(u)}\int_a^\infty \psi(t)\operatorname{Re}K_{1/2+iu}(pt)dt \tag{2.9}$$

and using the results obtained in Ref. 6, we reduce integral Eq. (2.7) to a Fredholm integral equation of the second kind in $\psi(t)$ of the form

$$\psi(t) + \int_a^\infty \psi(c)L(c,t)dc = R(t), \quad a < t < \infty$$

$$L(c,t) = \frac{4p}{\pi^2}\int_0^\infty [(\operatorname{sh}\pi u W^{-1}(u) - \operatorname{ch}\pi u)\operatorname{Re}K_{1/2+iu}(pc) -$$

$$- \operatorname{sh}\pi u B_u^*\{\operatorname{Re}K_{1/2+is}(pc)\}]\operatorname{Re}K_{1/2+iu}(pt)du$$

$$R(t) = \frac{\sqrt{2}p}{2\sqrt{\pi}G}\int_0^\infty \operatorname{sh}\pi u B_u\left\{\int_0^a q(r)K_{is}(pr)dr\right\}\operatorname{Re}K_{1/2+iu}(pt)du \tag{2.10}$$

It can be shown that the kernels of integral Eqs. (1.2), (2.1) and (2.10) are symmetrical. For sufficiently small values of $1 - 2\nu_1$, solutions of Fredholm integral equations of the second kind (1.4), (1.7), (2.3) and (2.5) can be constructed using successive approximations. The method of mechanical quadratures and tables of Bessel functions[8] can be used to solve these integral equations numerically.

The normal stress intensity factor

$$k_I = -\lim_{r\to a+0}\sqrt{r-a}\,q(r), \quad q(r) = 2G\int_0^a f(x)K(x,r)dx \ (r > a) \tag{2.11}$$

is expressed in terms of the solution of Eq.(2.10) by the formula

$$k_I = 4G\psi(a)/(\pi^2\sqrt{p}) \tag{2.12}$$

3. An elliptic cut

Suppose the cut occupies the elliptic region

$$\Omega = \{(r-c)^2/a^2 + z^2/b^2 \le 1\}, \quad b \ge a$$

For simplicity we will consider the case of a constant load $q(r, z) = q$. Separating the principal part of the kernel, we will rewrite Eq. (1.2) in dimensionless notation

$$r' = \frac{r-c}{b}, \quad x' = \frac{x-c}{b}, \quad z' = \frac{z}{b}, \quad y' = \frac{y}{b}, \quad \lambda = \frac{c}{b}, \quad \gamma = \frac{a}{b}, \quad q' = \frac{q}{2G},$$
$$f'(x', y') = \frac{f(x, y)}{bq'} \tag{3.1}$$

in the form (we henceforth omit the prime)

$$-\left(\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}\right)\int_{\Omega_1}\frac{f(x, y)}{R}dxdy + \int_{\Omega_1} f(x, y)L(x, y, r, z)dxdy = 2\pi, \quad (r, z) \in \Omega_1$$
$$L(x, y, r, z) = \frac{4}{\pi^2(r+\lambda)(x+\lambda)}\int_0^\infty\int_0^\infty \tau\,\mathrm{sh}\,\pi\tau K_{i\tau}(t(r+\lambda))[A_\tau\{sK_{is}(t(x+\lambda))\} -$$
$$-\tau\,\mathrm{cth}\,\pi\tau K_{i\tau}(t(x+\lambda))]\cos t(z-y)d\tau dt, \quad R = [(r-x)^2 + (z-y)^2]^{1/2} \tag{3.2}$$

where, corresponding to notation (3.1), $\Omega_1 = \{r^2/\gamma^2 + z^2 \le 1\}$.

We will use the regular asymptotic method[3,9] to solve integral Eq. (3.2). This method is effective for fairly large values of the dimensionless parameter λ, i.e., for a region Ω_1 fairly far from the edge of the wedge. It can be shown that the method can be used when $\lambda > \max(\gamma, \beta^{-1})$. For kernel (3.2) the following expansion holds ($\lambda \to \infty$)

$$L(x, y, r, z) = A\left(\frac{1}{\lambda^3} - \frac{3(x+r)}{2\lambda^4}\right) + O\left(\frac{1}{\lambda^5}\right)$$
$$A = \int_0^\infty \tau^2[\mathrm{th}\,\pi\tau W(\tau) - 1]d\tau + 32(1-\nu_1)\varepsilon\int_0^\infty \tau\,\mathrm{sh}\frac{\pi\tau}{2}W_0(\tau)\Psi(\tau)d\tau \tag{3.3}$$

Here, for Problems A and B

$$\Psi(\tau) = F^*(\tau)/W_1(\tau) \tag{3.4}$$

The function $F^*(\tau)$ when $\tau = \tau_n$ (at the n-th node of the quadrature formula) satisfies the following Fredholm integral equation of the second kind

$$F^*(\tau) - (1-2\nu_1)\int_0^\infty L_1(\tau, u)F^*(u)du = \tilde{F}(\tau), \; 0 \le \tau < \infty, \; \tilde{F}(\tau) = -\frac{\tau\,\mathrm{ch}(\pi\tau/2)W_0(\tau)}{\mathrm{ch}\,\pi\tau + \mathrm{ch}\,\pi\tau_n} \tag{3.5}$$

For Problem C

$$\Psi(\tau) = F_+^*(\tau)/W_+(\tau) - F_-^*(\tau)/W_-(\tau) \tag{3.6}$$

The functions $F_\pm^*(\tau)$ when $\tau = \tau_n$ are found from the system of Fredholm integral equations of the second kind ($0 \le \tau < \infty$)

$$F_\pm^*(\tau) - (1-2\nu_1)\int_0^\infty L_\pm(\tau, u)F_\pm^*(u)du = \pm\frac{(1-\nu_1)\varepsilon}{W_\mp(\tau)}W(\tau)F_\mp^*(\tau) + \frac{1}{2}\tilde{F}(\tau) \tag{3.7}$$

Taking expansion (3.3) into account, we obtain the asymptotic solution of Eq. (3.2) in the form

$$f(r, z) = \frac{\gamma}{E(e)}\sqrt{1 - \frac{r^2}{\gamma^2} - z^2}\left\{1 - \frac{A\gamma^2}{2\lambda^3}\left[\frac{2}{3E(e)} - \frac{e^2 r}{\lambda D(e)}\right] + O\left(\frac{1}{\lambda^5}\right)\right\}, \quad \lambda \to \infty$$
$$e = \sqrt{1-\gamma^2}, \quad D(e) = (1+e^2)E(e) + (1-e^2)K(e) \tag{3.8}$$

where $K(e)$ and $E(e)$ are complete elliptic integrals.

Table 1

β	α	A	B	C	A	B	C
			Values of A (3.3)			Values of $f(0, 0)$ (3.8)	
45°	5°	0.6040	1.667	−7.078	0.4107	0.4069	0.4380
	10°	(0.3016	1.160	(6.832	0.4139	0.4087	0.4371
	15°	(0.9874	0.5092	(6.385	0.4164	0.4111	0.4355
60°	5°	(0.04507	0.2417	(2.526	0.4130	0.4120	0.4218
	10°	(0.2896	0.1416	(2.485	0.4139	0.4124	0.4217
	15°	(0.4858	(0.007404	(2.398	0.4146	0.4129	0.4214

Table 2

θ	α	A	B	C
			Values of the NSIF (3.9)	
0°	5°	0.9959	0.9887	1.048
	10°	1.002	0.9922	1.046
	15°	1.007	0.9966	1.043
180°	5°	0.9937	0.9825	1.074
	10°	1.003	0.9879	1.071
	15°	1.010	0.9947	1.067

We obtain the following expression for the normal stress intensity factor (NSIF) on the cut contour

$$\frac{K_I}{K_I^\infty} = 1 - \frac{A\gamma^2}{2\lambda^3}\left[\frac{2}{3E(e)} - \frac{e^2\gamma\cos\theta}{\lambda D(e)}\right] + O\left(\frac{1}{\lambda^5}\right), \quad \lambda \to \infty \tag{3.9}$$

The angle θ is read off the r semiaxis and K_I^∞ is the NSIF for an elliptic cut in infinite space ($\lambda = \infty$).[10]

Calculations were carried out for Problems A, B and C with $\nu_1 = 0.5$ and $\varepsilon = 1$ (two similar materials; in view of the hinged coupling between the layers here this is not identical with the case of a uniform wedge). On the left-hand side of Table 1 we show values of the constant A (3.3), and on the right-hand side of Table 1 we show values of the opening of the crack $f(0, 0)$, calculated from formula (3.8) with $\lambda = 2$, $\gamma = 0.5$ and $\beta = 45°$ and 60°. For the same values of λ, γ and $\beta = 45°$, in Table 2 we give values of the NSIF (3.9). For Problem C (in view of the fact that there are no stresses on the outer surfaces of the composite wedge), the values of $f(0, 0)$ and NSIF are greater than for Problems A and B. At a point of the cut contour closest to the edge ($\theta = 180°$) compared with a point on the contour far from the edge of the wedge ($\theta = 0°$), the NSIF values are less for Problem B (due to the rigid clamping of the outer faces) and greater for Problem C.

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